Preparation for 3rd class-test in Stochastics

Karl Rupp

karlirupp@hotmail.com

Content

Toolbox:

- Generating functions
- Solving simple PDEs

Topics in Stochastics:

- Poisson Processes
- Branching Processes
- Population Dynamics
- Queues

Toolbox: Generating functions

Definition:

$$G_X(s) = \mathbb{E}[s^X] = \sum_{n=0}^{\infty} p_n s^n$$

of even time-dependent:

$$G_X(t,s) = \sum_{n=0}^{\infty} p_n(t) s^n$$

$p_n(t) \ldots$ represents the probability of an 'amount' of n (at time t)

Toolbox: Generating functions

Attention: Moment generating function is something different!

$$M_X(z) = \mathbb{E}[e^{zX}] = \sum_{n=0}^{\infty} p_n(t) e^{zn}$$

Always distinguish between $G_X(s)$ and $M_X(z)$!

Anyway, we won't need $M_X(z)$ in this seminar anymore, we will deal with $G_X(s)$.

Toolbox: Generating functions

Important properties:

1. Expectation:

$$\mathbb{E}[X] = \frac{\partial G(s,t)}{\partial s}\bigg|_{s=1}$$

2. Variance:

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 =$$

$$= \frac{\partial^2 G(s,t)}{\partial s^2}\bigg|_{s=1} + \frac{\partial G(s,t)}{\partial s}\bigg|_{s=1} - \left[\frac{\partial G(s,t)}{\partial s}\bigg|_{s=1}\right]^2$$

Take care: Do not forget about the second term!

Toolbox: Solving PDEs

Given a partial differential equation

$$\frac{\partial F(t,s)}{\partial t} = p(s)\frac{\partial F(t,s)}{\partial s}. \quad (1)$$

What to do about it? Try to transform to a much simpler PDE:

$$\frac{\partial Q(t,z)}{\partial t} = \frac{\partial Q(t,z)}{\partial z}, \quad (2)$$

whose solution is any function $w(z+t)$.

Toolbox: Solving PDEs

Let's try a tranformation $z = z(s)$, then $\frac{\partial F(t,s)}{\partial s}\frac{ds}{dz} = \frac{\partial F(t,z)}{\partial z}$ and therefore $\frac{ds}{dz} \stackrel{!}{=} p(s)$. Separation of variables leads to

$$\int \frac{ds}{p(s)} = \int dz = z + c. \tag{3}$$

The constant c can be set to 0. Inverting this expression leads to $s(z)$. We get

$$\frac{\partial F(t,z)}{\partial t} = \frac{\partial F(t,z)}{\partial z}. \tag{4}$$

Toolbox: Solving PDEs

- Now any function $w(z+t)$ solves (4).
- An initial condition $F(0,z) = w(z)$ fixes w (substitute $s(z)$ for s in $F(0,s)$).
- $F(t,z)$ is simply $w(z+t)$ (replace z by $z+t$)
- $F(t,s)$ can be restored by replacing z by $z(s)$. (See problem 3 of problem set 8)

That's it, our PDE is solved! :-)

Poisson Processes

Characterics of a Poisson process:
Probability that 'something' happens n times within the next t time units is given by

$$P_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}. \tag{5}$$

On the other hand: Time t_1, until 'something' happens has an exponential distribution:

$$f(t_1) = \lambda e^{-\lambda t_1} \tag{6}$$

Further details can be found in the lecture notes!

Branching processes: Discrete

Discrete branching process:
Two separate generating functions:

- $G(s) = \sum_{k=0}^{\infty} g_k s^k$ where g_k is the probability *for a single individuum* to have k descendants
- $F_j(s) = \sum_{k=0}^{\infty} P_{j,k} s^k$ for the whole population. $P_{j,k}$ is the probability to have a population of k in the j-th generation

There is a coupling between $G(s)$ and $F_j(s)$:

$$F_{j+1}(s) = F_j(G(s)) = G(F_j(s)).$$

Probability of extinction:

$$\min(\xi) : \xi = G(\xi),\ 0 \le \xi \le 1 \tag{7}$$

Branching processes

Continuous case:
Again: Two separate generating functions

- $G(t,s)$ for a single individuum (as in discrete case)
- $F(t,s)$ for whole population (as in discrete case)

Coupling between $G(t,s)$ and $F(t,s)$ becomes

$$F(t+dt,s) = F(t,G(s)).$$

Taylor expansion in dt finally leads to

$$\frac{\partial F(t,s)}{\partial t} = p(s)\frac{\partial F(t,s)}{\partial s}, \tag{8}$$

where $p(s)$ is the coefficient of dt in $G(s)$.

Branching processes

This PDE can be solved using our toolbox-method! :-)

The probability of extinction is simply the probability $P_0(t)$, which is the coefficient of s^0 in $F(t, s)$. Therefore:

$$P[\text{extinction}] = F(t, 0) \tag{9}$$

(Compare Problem no. 3 in problem set 8!)

Warning: In lecture notes $F(t, s)$ and $G(t, s)$ are sometimes mixed up!!

Population dynamics

The key to the solution are 'rate equations' and follow directly from continuous branching processes.

$$\frac{dP_j}{dt} = \lambda_{j-1}P_{j-1} - (\lambda_j + \mu_j)P_j + \mu_{j+1}P_{j+1}$$
$$\frac{dP_0}{dt} = \mu_1 P_1 - \lambda_0 P_0$$

Several possibilities for λ_j and μ_j!
Birth-Death: $\lambda_j = j\lambda$, $\mu_j = j\mu$,
Immigration-Emmigration: $\lambda_j = \lambda$, $\mu_{j>0} = \mu$, $\mu_0 = 0$

and many more ...

Population dynamics

How to solve them?

- Multiply j-th equation by s^j
- Sum all equations and use $F(t,s) = \sum_{j=0}^{\infty} P_j(t)s^j$
- Solve resulting PDE -> Toolbox!

Usually solving the PDE is not necessary! For equilibrium $\frac{\partial F(t,s)}{\partial t} = 0$, which transforms the PDE to an ODE that is (usually) easier to solve.

Queues

Our assumptions on a $M/M/1$-queue:

- Arrivals are a Poisson process
- Service time is exponentially distributed, $f(t) = \lambda e^{-\lambda t}$

Then a standard $M/M/1$-queue is equivalent to the immigration-emigration model with rates λ and μ. For the equilibrium one finds $P_j = \rho^j(1-\rho) = (\frac{\lambda}{\mu})^j(1-\frac{\lambda}{\mu})$.

More generally for the equilibrium:

$$P_j = \frac{\prod_{i=0}^{j-1}\lambda_i}{\prod_{i=1}^{j}\mu_i} P_0, \quad \text{and} \quad \sum_{j=0}^{\infty} P_j \stackrel{!}{=} 1 \tag{10}$$

Queues

Queue-characteristics:

- Traffic intensity: $\rho = \frac{\lambda}{\mu}$
- Mean no. of customers in the system: $\mathbb{E}[N] = \sum_{j=0}^{\infty} j P_j$.
- Mean no. of customers in the queue: $\mathbb{E}[N_q] = \sum_{j=1}^{\infty} (j-1) P_j$.
- Average System Processing Time (ASPT): $\mathrm{ASPT} = \sum_{j=0}^{\infty} \frac{j+1}{\mu} P_j$.

Queues

Some more queue-characteristics:

- Probability that server is free: P_0
- Average waiting time until customer arrives: $\mathbb{E}[\text{Slack}] = \frac{1}{\lambda}$
- Average busy period: $\mathbb{E}[\text{Busy}] = \mathbb{E}[\text{Slack}]\frac{1-P_0}{P_0}$

In that way all problems regarding queues can be solved! :-)

The End

Thank you for your attention
and good luck for the class test!