Fourier Transform and its Applications

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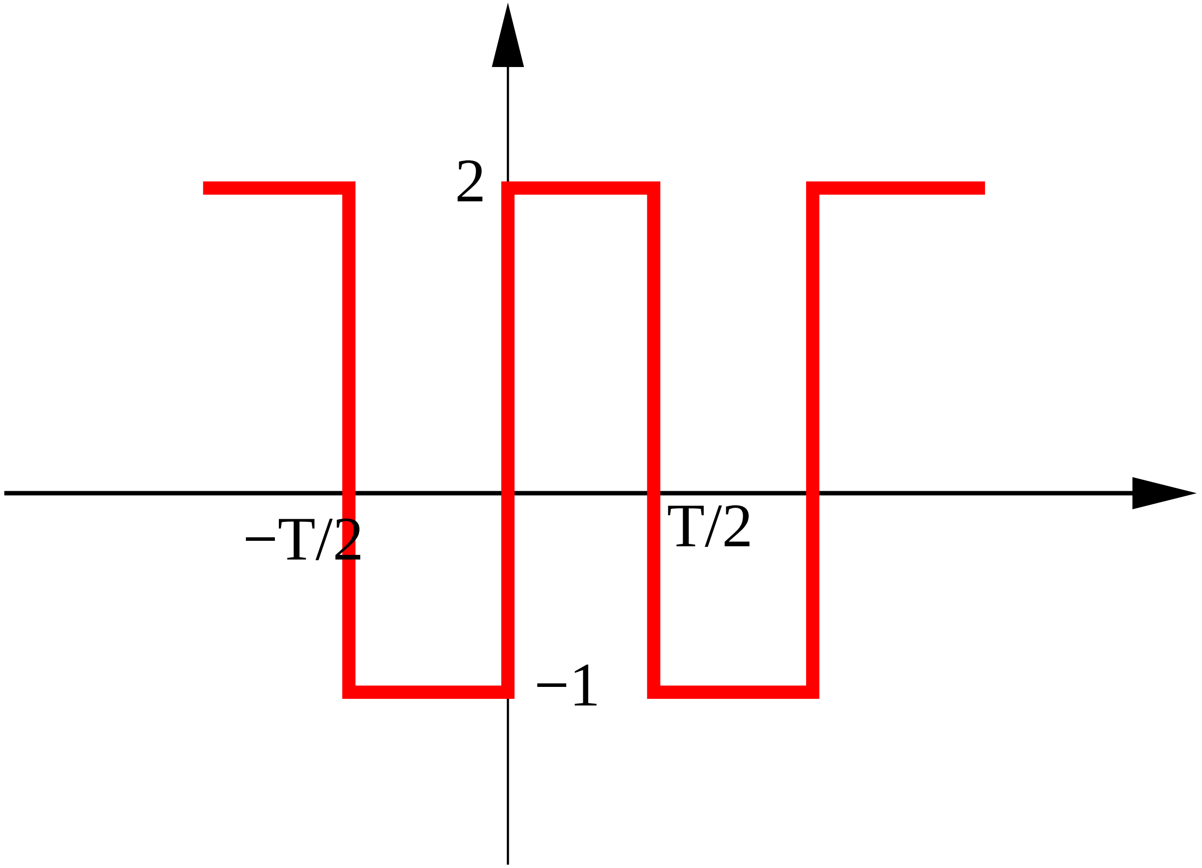
Motivation

Given any periodic signal $p(x)$:



Motivation II

- Decomposition into most basic types of periodic signals with same period: *Sine and Cosine*
- Candidates:

$$\sin(\frac{2\pi x}{T}), \sin(2\frac{2\pi x}{T}), \ldots$$
$$\cos(\frac{2\pi x}{T}), \cos(2\frac{2\pi x}{T}), \ldots$$

- Thus $p(x)$ could be rewritten as:

$$p(x) = \sum_{k=0}^{\infty} a_k \cos(k\frac{2\pi x}{T}) + b_k \sin(k\frac{2\pi x}{T})$$

Motivation III

An analogon:
Given a crowd of people from UK, France, Greece and from Germany. How to separate them?

(One possible) answer:

- Ask them to move on the left in French, forward in Greek, backwards in English and to move on the right in German.
- Use of spoken language as identifier.

Motivation IV

How to extract potions of sine and cosine?
$\Rightarrow$ A unique "identifier" for each sine and cosine needs to be found

Solution: Use scalar product, $k \in \mathbb{N}$:

$$\int_{-T/2}^{T/2} \cos(k\frac{2\pi x}{T}) \cos(n\frac{2\pi x}{T})dx = \begin{cases} T, & k = n = 0 \\ T/2, & k = n \neq 0 \\ 0, & k \neq n \end{cases}$$

Analogous results for $\sin(k\frac{2\pi x}{T}) \cdot \sin(n\frac{2\pi x}{T})$ and $\sin(k\frac{2\pi x}{T}) \cdot \cos(n\frac{2\pi x}{T})$!

Fourier series

Sticking all together leads to

$$p(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\frac{2\pi x}{T}) + b_k \sin(k\frac{2\pi x}{T})$$

with

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} p(x)\cos(k\frac{2\pi x}{T})dx, \qquad k \geq 0$$

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} p(x)\sin(k\frac{2\pi x}{T})dx, \qquad k \geq 1$$

Fourier series II

Simplification using $e^{ix} = \cos(x) + i\sin(x)$:

$$p(x) = \sum_{k=-\infty}^{\infty} c_k e^{i\frac{2\pi x}{T}}$$

with

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} p(x) e^{i\frac{2\pi x}{T}} dx, \quad k \geq 0$$

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- Coefficients not only at discrete values, but defined over the whole real axis.
- Fourier transform becomes an operator (function in - function out)
- Periodicy of function not necessary anymore, therefore arbitrary functions can be transformed!

Fourier transform

Fourier transform in one dimension:

$$\mathcal{F}\{f\}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

Can easily be extended to several dimensions:

$$\mathcal{F}\{f\}(\omega) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-i\omega \mathbf{x}} d\mathbf{x}$$

Often capital letters are used for the Fourier transform of a function. ($f(x) \Longleftrightarrow F(\omega)$)

Basic Properties

- Duality: $\mathcal{F}\{\mathcal{F}\{f\}\}(x) = f(-x)$
  or more often used:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega$$

- Linearity: $a \cdot f(x) + b \cdot g(x) \Longleftrightarrow a \cdot F(\omega) + b \cdot G(\omega)$
- Scaling: $f(a \cdot x) \Longleftrightarrow \frac{1}{|a|} F(\frac{x}{a})$
- Shift in f: $f(x-a) \Longleftrightarrow e^{-iax} F(\omega)$
- Shift in F: $e^{iax} f(x) \Longleftrightarrow F(\omega - a)$

Further Properties

- Differentiation of f: $\frac{d^n f(x)}{dx^n} \Longleftrightarrow (i\omega)^n F(\omega)$
- Differentiation of F: $x^n f(x) \Longleftrightarrow i^n \frac{d^n G(\omega)}{d\omega}$
- Convolution of f, g: $f(x) * g(x) \Longleftrightarrow F(\omega)G(\omega)$
- Convolution of F, G: $f(x)g(x) \Longleftrightarrow \frac{F(\omega)*G(\omega)}{\sqrt{2\pi}}$
- Parseval theorem:

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \int_{-\infty}^{\infty} F(\omega)\overline{G(\omega)}d\omega$$

Some Fourier pairs

Some of the most important transform-pairs:

$$\text{rect}(x) \Longleftrightarrow \frac{2}{\sqrt{2\pi}} \frac{\sin(\omega/2)}{\omega}$$

$$\delta(x) \Longleftrightarrow \frac{1}{\sqrt{2\pi}}$$

$$e^{-\alpha t} \Longleftrightarrow \frac{1}{\sqrt{2\alpha}} \cdot e^{-\frac{\omega^2}{4\alpha}}$$

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) \Longleftrightarrow \frac{\sqrt{2\pi}}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

Making use of Fourier transform

- Differential equations transform to algebraic equations that are often much easier to solve

- Convolution simplifies to multiplication, that is why Fourier transform is very powerful in system theory

- Both $f(x)$ and $F(\omega)$ have an "intuitive" meaning

Discrete Fourier Transform (DFT)

The power of Fourier transform works for digital signal processing (computers, embedded chips) as well, but of course a discrete variant is used (notation applied to conventions):

$$X(k) = \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i}{N}kn} \quad k = 0, \ldots, N-1$$

for a signal of length N.

Dirac-Delta-Function (discrete)

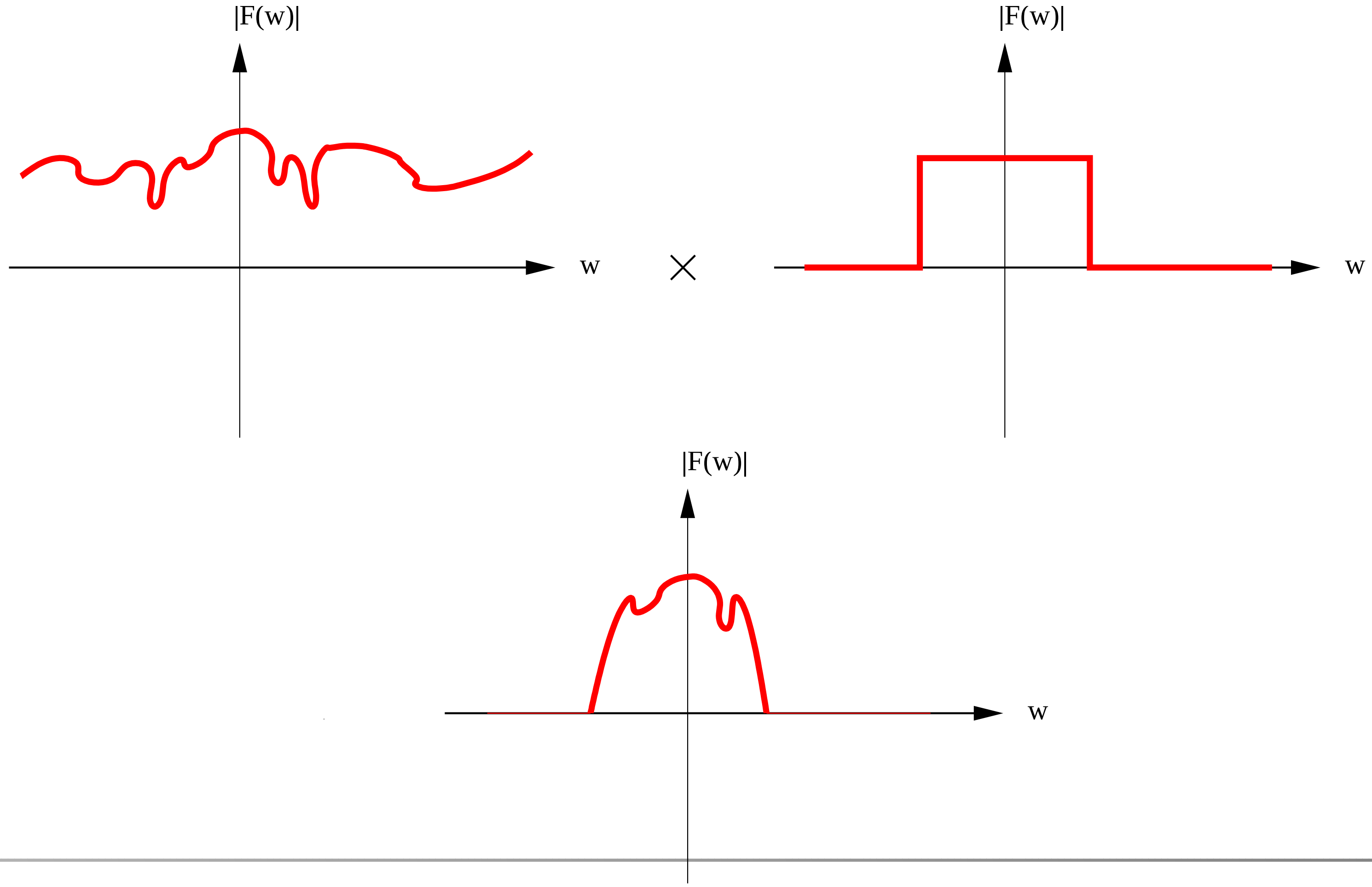
The Delta-distribution in terms of digital systems is simply defined as

$$x(n) = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

(Input-)signals are decomposed into such delta-functions, while the output is a superposition of the output for each of the input-delta-functions.

Application I

Filtering audio



Application II

Partial Differential Equations:

Find bounded solutions $u(x,t)$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$

$$\frac{\partial^2}{\partial t^2} u(x,t) + \Delta_x u(x,t) = 0$$
$$u(x,0) = f(x)$$

Solution: Using Fourier transform with respect to x.

$$u(x,t) = \pi^{-\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) \int_{\mathbb{R}^n} f(y) \frac{t}{(t^2 + |x-y|^2)^{\frac{n+1}{2}}} dy.$$

Functional Analysis View

- Integral operations well defined for $f \in L_1(\mathbb{R}^n)$ (Fubini).
- But where is Fourier-transform continuous?
- Is it one-to-one?

Starting with test-functions: They are not enough. Hence: *Rapidly decreasing functions* $\mathcal{S}_n$

$$f \in C^\infty(\mathbb{R}^n) : \sup_{|x|<N} \sup_{x\in\mathbb{R}^n} (1+|x|^2)^N \left|\frac{\partial^\alpha f(x)}{\partial x^\alpha}\right| < \infty$$

for $N = 0, 1, 2, \ldots$ and for multi-indices α.

Rapidly decreasing functions

- Form a vector space
- Fourier transform is a continuous, linear, one-to-one mapping of $\mathcal{S}_n$ onto $\mathcal{S}_n$ of period 4, with a continuous inverse.
- Test-functions are dense in $\mathcal{S}_n$
- $\mathcal{S}_n$ is dense in both $L_1(\mathbb{R}^n)$ and $L_2(\mathbb{R}^n)$
- Plancharel theorem: There is a linear isometry of $L_2(\mathbb{R}^n)$ onto $L_2(\mathbb{R}^n)$ that is uniquely defined via the Fourier transform in $\mathcal{S}_n$.

Extensions

- Fast Fourier Transform (FFT): effort is only $O(n \log(n))$ instead of $O(n^2)$
- Laplace transform:

$$F(s) = \int_{0-}^{\infty} f(x) e^{-sx} dx$$

- z-transform: Discrete counterpart of Laplace transform

$$X(z) = Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

The End

Thank you for your attention!